

Characterizing meromorphic psuedo-lemniscates

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Abstract

Let f be a meromorphic function with simply connected domain $G \subset \mathbb{C}$, and let $\Gamma \subset \mathbb{C}$ be a smooth Jordan curve. We call a component of $f^{-1}(\Gamma)$ in G a Γ -pseudo-lemniscate of f . In this note we give criteria for a smooth Jordan curve S in G (with bounded face D) to be a psuedo-lemniscate of f in terms of the number of preimages (counted with multiplicity) which a given w has under f in D , as w ranges over the Riemann sphere. We also develop a test, in the same terms, by which one may show that the image of a Jordan curve under f is not a Jordan curve.

1 Introduction

It is well known that a degree n Blaschke product is an n -to-1 (counting multiplicity) analytic self-map of the unit disk \mathbb{D} , and moreover that every such analytic n -to-1 self-map of the unit disk is a degree n Blaschke product. We can restate this in a slightly more general setting. Throughout this paper we let $G \subset \mathbb{C}$ denote a simply connected domain, and we let $S \subset G$ be a smooth Jordan curve, with bounded face D .

Theorem 1. *Let $f : G \rightarrow \mathbb{C}$ be analytic. The following are equivalent.*

1. *S is a lemniscate of f in G .*
2. *For some positive integer n , f maps D precisely n -to-1 onto \mathbb{D} .*

If Items 1 and 2 hold, the function $f \circ \phi$ is a degree n Blaschke product, where $\phi : \mathbb{D} \rightarrow D$ is any Riemann map.

The purpose of this note is to generalize the result of Theorem 1 to the context of meromorphic functions and pseudo-lemniscates. In order to state the theorem, we begin with several definitions. For the remainder of the paper we let Γ denote some smooth Jordan curve in \mathbb{C} , with bounded face Ω_- and unbounded face Ω_+ (where Ω_+ includes the point ∞).

Definition.

- *Let $f : G \rightarrow \hat{\mathbb{C}}$ be meromorphic. Suppose that S is a component of $f^{-1}(\Gamma)$. We call S a Γ -pseudo-lemniscate of f .*

- For any $w \in \hat{\mathbb{C}}$, let $\mathcal{N}_f(w)$ denote the number of preimages under f of w in D , counted with multiplicity.

In the special case that $\Gamma = \mathbb{T}$, the above definition reduces to the classical definition of a lemniscate. The definition of a pseudo-lemniscate was introduced in [1] wherein conformal equivalence of functions on domains bounded by pseudo-lemniscates was studied. We now proceed to the promised generalization of Theorem 1.

Theorem 2. *Let $f : G \rightarrow \hat{\mathbb{C}}$ be meromorphic. The following are equivalent.*

1. \mathcal{S} is a Γ -pseudo-lemniscate of f in G .
2. For some non-negative integers n_- and n_+ , the following holds.
 - For every $w \in \Omega_-$, $\mathcal{N}_f(w) = n_-$.
 - For every $w \in \Omega_+$, $\mathcal{N}_f(w) = n_+$.
 - For every $w \in \Gamma$, $\mathcal{N}_f(w) = \min(n_-, n_+)$.

If Items 1 and 2 hold, and f is analytic in D , then $n_+ = 0$, and the function $\psi^{-1} \circ f \circ \phi$ is a degree n_- Blaschke product, where $\phi : \mathbb{D} \rightarrow D$ and $\psi : \mathbb{D} \rightarrow \Omega_-$ are any Riemann maps.

If Items 1 and 2 hold, and $\Gamma = \mathbb{T}$, then $f \circ \phi$ is a ratio of Blaschke products (degree n_- in the numerator and degree n_+ in the denominator), where $\phi : \mathbb{D} \rightarrow D$ is any Riemann map.

Using the fact that any meromorphic function is conformal away from its critical points, we may use Theorem 2 to supply a test for when the image of \mathcal{S} under f will not be a Jordan curve.

Corollary 3. *Let $f : G \rightarrow \hat{\mathbb{C}}$ be meromorphic. If there are three points $w_1, w_2, w_3 \in \hat{\mathbb{C}}$ such that the numbers $\mathcal{N}_f(w_1)$, $\mathcal{N}_f(w_2)$, and $\mathcal{N}_f(w_3)$ are distinct, then either f has a critical point on \mathcal{S} or $f(\mathcal{S})$ is not a Jordan curve.*

In Section 2 we will provide proofs of Theorem 2 and Corollary 3. In Section 3 we apply Corollary 3 to show that the image of a certain smooth Jordan curve \mathcal{S} under a certain meromorphic function f is not a Jordan curve, without explicitly finding a point of intersection in $f(\mathcal{S})$.

2 Lemmas and Proofs

We begin with a topological observation on the convergence of preimages under continuous maps. First we introduce a bit of standard notation.

Definition. For any $x \in \mathbb{C}$, $X \subset \mathbb{C}$, and $\delta > 0$, we define

$$B(x; \delta) = \{y \in \mathbb{C} : |x - y| < \delta\}$$

and

$$B(X; \delta) = \bigcup_{z \in X} B(z; \delta).$$

Lemma 4. *Let $U \subset \mathbb{C}$ be an open bounded set, and let $h : U \rightarrow \hat{\mathbb{C}}$ be continuous. Choose some $w_0 \in \mathbb{C}$. Then for any $\delta > 0$ there exists an $\epsilon > 0$ such that if $w \in B(w_0; \epsilon)$, then $h^{-1}(w) \subset B(h^{-1}(w_0) \cup \partial U; \delta)$.*

Proof. This is an elementary exercise in compactness and continuity. \square

Our second lemma shows that, under the conditions of Item 2 from Theorem 2, if E is a component of $f^{-1}(\Omega_-) \cap D$, and $z \in \partial E$, then $f(z) \in \partial \Omega_-$.

Lemma 5. *Let $h : G \rightarrow \hat{\mathbb{C}}$ be meromorphic. Let $\Omega \subset \mathbb{C}$ be open, bounded, and connected, and assume that the quantity $\mathcal{N}_h(w)$ is independent of the choice of $w \in \Omega$. Then if $E \subset D$ is a component of the set $h^{-1}(\Omega) \cap D$, and $z_0 \in \partial E$, then $h(z_0) \in \partial \Omega$.*

Proof. Let h , Ω , and E be as in the statement of the lemma. Choose some $z_0 \in \partial E$, and set $w_0 = h(z_0)$. Suppose by way of contradiction that $w_0 \in \Omega$. We proceed by cases.

Case 1. $z_0 \in D$.

Since h is continuous and Ω and D are open, there is a neighborhood of z_0 contained in $h^{-1}(\Omega) \cap D$, contradicting our choice of z_0 .

Case 2. $z_0 \notin D$.

Set $m = \mathcal{N}_h(w_0)$, and let z_1, z_2, \dots, z_k be the distinct preimages of w_0 in D , having multiplicities m_1, m_2, \dots, m_k (so that $m = m_1 + m_2 + \dots + m_k$). Choose some $\delta > 0$ such that $\delta < \min(|z_i - z_0| : 1 \leq i \leq k)$. Choose disjoint neighborhoods $D_1, D_2, \dots, D_k \subset D$ such that for each $1 \leq i \leq k$, the following holds.

- $z_i \in D_i$.
- $D_i \cap B(z_0; \delta) = \emptyset$.
- h is exactly m_i -to-1 on $D_i \setminus \{z_i\}$.

Choose some point $x \in B(z_0; \delta) \cap E$ sufficiently close to z_0 so that

$$h(x) \in \bigcap_{i=1}^k h(D_i).$$

Then $h^{-1}(h(x)) \cap D$ contains x , along with m_i distinct points in D_i , for each $1 \leq i \leq k$. Thus since $x \notin D_i$ for any $1 \leq i \leq k$, we have that $\mathcal{N}_h(h(x))$ is at least

$$m_1 + m_2 + \dots + m_k + 1 > m.$$

Since $x \in E$, $h(x) \in \Omega$, so we have a contradiction of the choice of m .

We therefore conclude that $w_0 \notin \Omega$. Since h is continuous we must have that $w_0 \in \overline{\Omega}$, so we conclude that $w_0 \in \partial \Omega$. \square

With these lemmas under our belt, we proceed to the proof of Theorem 2.

Proof of Theorem 2. We begin by assuming that \mathcal{S} is a Γ -pseudo-lemniscate of f in G .

Fix now some component E of $f^{-1}(\Omega_-)$ in D . We wish to show that there is some positive integer n_E such that f maps E exactly n_E -to-1 onto Ω_- (counting multiplicity).

Let $\Lambda_1, \Lambda_2, \dots, \Lambda_k$ denote the components of ∂E , with Λ_1 being the component of ∂E which is in the unbounded component of E^c . Since f is continuous and open, f maps each Λ_i to the boundary of Ω_- , namely to Γ . Therefore for each i , we define n_i to be the number of times $f(z)$ winds around Γ as z traverses Λ_i one time with positive orientation.

By the Riemann mapping theorem, for any point $w \in \Omega_-$, we can find some conformal map $\rho : \Omega_- \rightarrow \mathbb{D}$ such that $\rho(w) = 0$, and since the boundary of Ω_- is smooth, each such ρ extends conformally to a neighborhood of $\overline{\Omega_-}$. Therefore, replacing f with $\rho \circ f$, the components of ∂E are honest lemniscates of $\rho \circ f$. Since ρ is a conformal map on the closure of Ω_- , if $f(z)$ winds once around Γ , $\rho \circ f(z)$ winds once around the unit circle, and with the same orientation (since Ω_- is simply connected and ρ is analytic, ρ cannot reverse the orientation).

Therefore it follows from the argument principle that the number of zeros of $\rho \circ f$ in E (counted with multiplicity) is

$$n_E := n_1 - (n_2 + n_3 + \dots + n_k).$$

That is, the number of members of $f^{-1}(w)$ in E (counted with multiplicity) is n_E . Since this number n_E is independent of the choice of $w \in \Omega_-$, we conclude that f maps E precisely n_E -to-1 onto Ω_- . Therefore for any $w \in \Omega_-$,

$$\mathcal{N}_f(w) = \sum_E n_E.$$

This sum is taken over all components E of $f^{-1}(\Omega_-)$ in D , a quantity which again is independent of the choice of $w \in \Omega_-$. We call this sum n_- . Doing the same calculation with Ω_- replaced by Ω_+ , we obtain the corresponding number n_+ .

We now consider the set $f^{-1}(\Gamma) \cap D$. Since \mathcal{S} is a connected component of $f^{-1}(\Gamma)$ and \mathcal{S} is compact, and $f^{-1}(\Gamma)$ is closed in G , it follows that $f^{-1}(\Gamma) \setminus \mathcal{S}$ is isolated from \mathcal{S} by some positive distance. Let $\delta > 0$ be chosen to be less than half that distance. That is,

$$\inf_{x \in \mathcal{S}, y \in f^{-1}(\Gamma) \setminus \mathcal{S}} |x - y| > 2\delta.$$

Since D is simply connected, the set $D_\delta := \{z \in D : d(z, \mathcal{S}) < \delta\}$ is connected (where $d(z, \mathcal{S})$ denotes the distance $\inf_{x \in \mathcal{S}} |z - x|$). Therefore since f is continuous, and $f^{-1}(\Gamma)$ does not intersect D_δ , we conclude that either $f(D_\delta) \subset \Omega_-$ or $f(D_\delta) \subset \Omega_+$. Without loss of generality assume that $f(D_\delta) \subset \Omega_-$.

Choose now some w_0 in Γ . Let $z_1, z_2, \dots, z_l \in D$ denote the distinct preimages of w_0 in D , having multiplicities m_1, m_2, \dots, m_l . Choose disjoint domains $D_1, D_2, \dots, D_l \subset D$ such that the following holds for each i .

- $z_i \in D_i$.
- $D_i \cap D_\delta = \emptyset$.
- f has no critical points in $D_i \setminus \{z_i\}$.
- f is exactly m_i -to-1 in $D_i \setminus \{z_i\}$.

Reduce now $\delta > 0$ if necessary so that for each i , $B(z_i; \delta) \subset D_i$. By Lemma 4 we can find an $\epsilon > 0$ such that if $w \in \Omega_+$ is within ϵ of w_0 , then

$$f^{-1}(w) \cap D \subset B((f^{-1}(w_0) \cap D) \cup \partial D; \delta) \cap D.$$

Choose $\epsilon > 0$ smaller if necessary so that for each i , $B(w_0; \epsilon) \subset f(D_i)$.

Fix now some $w \in B(w_0; \epsilon) \cap \Omega_+$. Since $w \in \Omega_+$, and $f(D_\delta) \subset \Omega_-$, no point in $f^{-1}(w) \cap D$ is within δ of ∂D . Therefore by choice of ϵ we have

$$f^{-1}(w) \cap D \subset B(f^{-1}(w_0) \cap D; \delta) \subset \bigcup_{i=1}^l D_i.$$

Now, for each i , since $B(w_0; \epsilon) \subset f(D_i)$, $w \in f(D_i)$. Since f is exactly m_i -to-1 in $D_i \setminus \{z_i\}$, there are exactly m_i distinct points in $f^{-1}(w) \cap D_i$, and since none of these points are critical points, each one contributes exactly 1 to the total n_+ many elements in $f^{-1}(w) \cap D$. We therefore conclude that $\mathcal{N}_f(w_0) = n_+$.

Note that if we had selected the point w to be a point in Ω_- instead of in Ω_+ , we would have obtained the same result, except that some members of $f^{-1}(w) \cap D$ could additionally reside in D_δ . Our conclusion would then be that the $\mathcal{N}_f(w_0) \leq n_-$. Since $\mathcal{N}_f(w_0) = n_+$ and $\mathcal{N}_f(w_0) \leq n_-$, we therefore have that

$$\mathcal{N}_f(w_0) = \min(n_-, n_+).$$

If we had adopted the other assumption, that $f(D_\delta) \subset \Omega_+$, then similar work would give the same result. This concludes the first direction of our proof.

Suppose now that the second item holds. That is, that there are some non-negative integers n_- and n_+ for which the following holds.

- For every $w \in \Omega_-$, $\mathcal{N}_f(w) = n_-$.
- For every $w \in \Omega_+$, $\mathcal{N}_f(w) = n_+$.
- For every $w \in \Gamma$, $\mathcal{N}_f(w) = \min(n_-, n_+)$.

It follows from Lemma 5 that if E is any component of $f^{-1}(\Omega_-) \cap D$ or of $f^{-1}(\Omega_+) \cap D$, then $f(\partial E) \subset \Gamma$. We therefore conclude that $f(\mathcal{S}) = f(\partial D) \subset \Gamma$. Our goal is to show that \mathcal{S} itself is an entire component of $f^{-1}(\Gamma)$ in G .

For any point $z \in \mathcal{S}$, since f is analytic at z and Γ is smooth, if z is not a critical point of f then $f^{-1}(\Gamma)$ is a smooth curve close to z . On the other hand, if z is a critical point of f with multiplicity m then, close to z , $f^{-1}(\Gamma)$ consists of $2(m+1)$ paths meeting at z , forming equal angles $\frac{2\pi}{2(m+1)}$ around z . Since \mathcal{S} is smooth, it therefore suffices to show that our assumption on f implies that f does not have any critical points on \mathcal{S} .

Suppose by way of contradiction that $z_0 \in \mathcal{S}$ is a critical point of f with multiplicity m . Let Λ denote the component of $f^{-1}(\Gamma)$ which contains \mathcal{S} . Since $m \geq 1$,

$$\frac{2\pi}{2(m+1)} = \frac{\pi}{m+1} < \pi.$$

Therefore since \mathcal{S} is smooth, there are edges of Λ which extend from z_0 into either face of \mathcal{S} , and in particular into D . Let γ denote some such edge of Λ emanating from z_0 into D . Assume without loss of generality that as z traverses γ , $f(z)$ traverses Γ with positive orientation. Then for any point $w \in \Gamma$ taken arbitrarily close to $f(z_0)$ (with positive orientation around Γ from $f(z_0)$), there is some point $z \in D$ (on γ) which is arbitrarily close to z_0 , such that $f(z) = w$. A contradiction now follows from exactly the same reasoning as was used to prove Lemma 5.

We therefore conclude that \mathcal{S} contains no critical point of f , and thus that the component of $f^{-1}(\Gamma)$ which contains \mathcal{S} is just \mathcal{S} itself. That is, \mathcal{S} is a Γ -pseudo-lemniscate of f in G . This concludes the second direction of the proof.

Now suppose that Items 1 and 2 in the theorem do obtain, and that moreover f is analytic in D . Since $f^{-1}(\infty) \cap D = \emptyset$, immediately we have that $n_+ = 0$. If $\phi : \mathbb{D} \rightarrow D$ and $\psi : \mathbb{D} \rightarrow \Omega_-$ are Riemann maps, then $\psi^{-1} \circ f \circ \phi$ is a n_- -to-1 analytic self map of \mathbb{D} , which by Theorem 1 is a degree n_- Blaschke product.

Finally, suppose that Items 1 and 2 in the theorem do obtain, and that $\Gamma = \mathbb{T}$. Let $\phi : \mathbb{D} \rightarrow D$ be any Riemann map. Since ∂D is smooth, ϕ extends conformally to a neighborhood of the closed unit disk. Since f satisfies Item 2, $f \circ \phi$ has n_- zeros and n_+ poles in \mathbb{D} , and since f satisfies Item 1, $|f \circ \phi| = 1$ on \mathbb{T} . Let A denote a degree n_- Blaschke product having the same zeros as $f \circ \phi$, with the same multiplicities, and let B denote a degree n_+ Blaschke product, having zeros exactly at the locations of the poles of $f \circ \phi$, with the same multiplicities. Then $(f \circ \phi) \cdot \frac{B}{A}$ has no zeros or poles in \mathbb{D} and on ∂D ,

$$\left| (f \circ \phi) \cdot \frac{B}{A} \right| = |f \circ \phi| \cdot \frac{|B|}{|A|} = 1.$$

Therefore by the maximum/minimum modulus theorems, $(f \circ \phi) \cdot \frac{B}{A}$ is a unimodular constant λ . Replacing A by $\lambda \cdot A$, we have $f \circ \phi = \frac{A}{B}$. \square

Proof of Corollary 3. Suppose that f , w_1 , w_2 , and w_3 are as in the statement of the lemma. Suppose that $f(\mathcal{S})$ is a Jordan curve. Setting $\Gamma = f(\mathcal{S})$, since $\mathcal{N}_f(w_1)$, $\mathcal{N}_f(w_2)$, and $\mathcal{N}_f(w_3)$ are distinct, \mathcal{S} cannot be a Γ -pseudo-lemniscate

of f by Theorem 2. Let Λ denote the component of $f^{-1}(\Gamma)$ in G which contains \mathcal{S} . Since $\Lambda \neq \mathcal{S}$, Λ must branch off from \mathcal{S} at some point, and thus f must not be conformal at that point. That is, f must have a critical point on \mathcal{S} . \square

3 Application of Corollary 3

Example. Let \mathcal{S} denote some smooth curve in \mathbb{C} which contains the quadrilateral with vertices 0, 1, $2i$, and $1 + 4i$, and which approximates the boundary of this shape closely. Let D denote the bounded face of \mathcal{S} . Then setting $f(z) = e^z$, we claim that $f(\mathcal{S})$ is not a Jordan curve.

To see this, observe that for $w_1 = e^0i$, $w_2 = e^1i$, and $w_3 = e^2i$, we have $\mathcal{N}_f(w_1) = 1$, $\mathcal{N}_f(w_2) = 2$, and $\mathcal{N}_f(w_3) = 0$. Therefore Corollary 3 tells us that either f has a critical point on \mathcal{S} or $f(\mathcal{S})$ is not a Jordan curve. Since $f(z) = e^z$ has no critical points, we conclude that $f(\mathcal{S})$ is not a Jordan curve.

References

- [1] T. Richards and M. Younsi, Conformal Models and Fingerprints of Pseudo-lemniscates, *Const. Approx.*, <http://dx.doi.org/10.1007/s00365-016-9348-0> (2016), 1–13.